ON THE REDUCTION OF CERTAIN INTEGRAL EQUATIONS OF THE FIRST KIND IN THE THEORY OF ELASTICITY AND HYDRODYNAMICS TO EQUATIONS OF THE SECOND KIND

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It is shown here that the solution of certain integral equations of the first kind which are encountered in applications reduce to solution of two equations: 1) the simplest equation of the first kind, the Abel integral equation; and 2) the equation of the second kind.

1. We shall examine the equation of the theory of a slender wing ([1], p. 80)

$$\int_{\alpha_0}^{\alpha_1} u(\alpha)\left(\frac{2a+\alpha_1-\alpha}{\alpha_1-\alpha}\right)^{1/2} d\alpha = \int_{-a}^{a} (v_{n0}-v_n)\left(\frac{a-\xi}{a+\xi}\right)^{1/2} d\xi \qquad \begin{pmatrix}\alpha_1>\alpha_0\\ a>0\end{pmatrix} \tag{1.1}$$

with the unknown function $u(\alpha)$. For $a=\text{const}$, the kernel of this equation depends only on the difference of the arguments, and Eq. (1.1) can be solved in closed form with the aid of Mellin's formula [2]; however, as noted in [1], this approach is not effective because of a computational difficulty.

Replacing the right-hand side of (1.1) by an arbitrary function, we rewrite (1.1) in the form

$$\int_{\alpha_0}^{x} u(\tau)\left(\frac{2a+x-\tau}{x-\tau}\right)^{1/2} d\tau = f(x, a) \qquad (x>\alpha_0) \tag{1.2}$$

or

$$\sqrt{2a}\int_{\alpha_0}^{x}\frac{u(\tau)\,d\tau}{\sqrt{x-\tau}} = \int_{\alpha_0}^{x}\frac{\sqrt{2a}-\sqrt{2a+x-\tau}}{\sqrt{x-\tau}}\,u(\tau)\,d\tau + f(x, a) \tag{1.3}$$

Inverting (1.3) as an Abel equation we have

$$u(x) + \frac{1}{\pi\sqrt{2a}}\frac{d}{dx}\int_{\alpha_0}^{x}\frac{d\tau}{\sqrt{x-\tau}}\int_{\alpha_0}^{\tau}\frac{\sqrt{2a+\tau-\xi}-\sqrt{2a}}{\sqrt{\tau-\xi}}\,u(\xi)\,d\xi = f_1(x, a) \tag{1.4}$$

$$f_1(x, a) = \frac{1}{\pi\sqrt{2a}}\frac{d}{dx}\int_{\alpha_0}^{x}\frac{f(\tau, a)}{\sqrt{x-\tau}}\,d\tau$$

Changing the order of integration in (1.4) and differentiating under the integral sign, we obtain

$$u(x) + \frac{1}{4a\pi}\int_{\alpha_0}^{x} M\left(\frac{x-\xi}{2a}\right) u(\xi)\,d\xi = f_1(x, a)$$

$$M\left(\frac{x-\xi}{2a}\right) = 4a\frac{d}{dx}\int_{\xi}^{x}\left[\left(1-\frac{\tau-\xi}{2a}\right)^{1/2}-1\right]\frac{d\tau}{\sqrt{x-\tau}\,\sqrt{\tau-\xi}} = \tag{1.5}$$

$$= \int_0^1 \left(\frac{s}{1-s}\right)^{1/2} \frac{ds}{\sqrt{1+{}^1/_2 s(x-\xi)/a}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{x-\xi}{2a}\right) \qquad (1.6)$$

The Volterra Eq. (1.5) has a kernel of difference type and can, therefore, be solved in closed form like the original equation. However, unlike (1.2), the solution of Eq. (1.5) can be obtained simply by the method of successive approximations.

We note that the indicated method of reduction of Eq. (1.2) to Eq. (1.5) is also applicable in the case $a \neq$ const, but then the kernel of Eq. (1.5) will not depend only on the difference $(x-\xi)$.

By means of the change of variables in (1.6)

$$s = 1 - \frac{y^2}{k^2}, \quad k = \left(\frac{z}{1+z}\right)^{1/2} = \left(\frac{x-\xi}{2a+x-\xi}\right)^{1/2} \qquad (0<k<1)$$

the kernel $M(z)$, $z=\frac{1}{2}(x-\xi)/a$ can be reduced to the form

$$M(z) = \frac{2}{\sqrt{z}} \int_0^k \left[\frac{1}{1-y^2}\left(1-\frac{y^2}{k^2}\right)\right]^{1/2} dy = \frac{2}{\sqrt{z}} E\left(\sin^{-1} k, \frac{1}{k}\right)$$

where $E(\varphi, 1/k)$ is the elliptic integral of the second kind. We may also express $M(z)$ in terms of complete elliptic integrals of the first and second kinds

$$M(z) = \frac{2\sqrt{1+z}}{z}\left[E(k) - \frac{K(k)}{1+z}\right]$$

(see 8.111, 8.112 or 3.169, 9 in [9]).

2. The method of separation of the Abelian part of an equation which has been illustrated by the example of Eq. (1.2) can be generalized to other equations. Besides the trivial generalization to the case of the kernel

$$\left(\frac{2a+x-\tau}{x-\tau}\right)^{\mu} \qquad (0<\mu<1)$$

in (1.2), this method can be successfully applied to Eq.

$$\int_\alpha^\beta \frac{c_1 + c_2 \operatorname{sign}(x-\tau)}{|x-\tau|^{\mu}} \varphi(\tau) d\tau + T\varphi = f(x) \qquad (2.1)$$

where c_1 and c_2 are functions which depend only on x or only on τ, and the kernel of the integral operator T either has no singularity whatever on the diagonal $x=\tau$, or else has a singularity of power type with exponent smaller than μ.

Various special cases of this equation (which for $T=0$ has been called the generalized Abel equation [3]) have important applications in the theory of elasticity [4], the theory of creep and the theory of plasticity [5 to 7]. Equation (2.1) with $T=0$ was first solved by Sakaliuk [3 and 8] by the method of analytic continuation. This equation was investigated by the author of the present paper using another method. We indicate here only that by separating the Abelian part

$$\int_\alpha^x \frac{c_1+c_2}{(x-\tau)^{\mu}} \varphi(\tau)\, d\tau$$

out of (2.1) and applying to (2.1) the operator $I_{\alpha x}^{-(1-\mu)}$, which is the inverse of the operator

$$I_{\alpha x}{}^{1-\mu}\varphi \equiv \frac{1}{\Gamma(1-\mu)}\int_{\alpha}^{x}\frac{\varphi(\tau)\,d\tau}{(x-\tau)^{\mu}}$$

we reduce Eq. (2.1) to a singular equation with a kernel of Cauchy type. The operator $I_{\alpha x}^{-(1-\mu)}$ must be applied to (2.1) on the left if the coefficients c_1 and c_2 depend on τ and on the right if they depend on x. The following identities have been used:

$$I_{\alpha x}{}^{-\nu}I_{\tau\beta}{}^{\nu}\varphi \equiv \cos(\nu\pi)\,\varphi(x) + \frac{\sin\nu\pi}{\pi}\frac{1}{(x-a)^{\nu}}\int_{\alpha}^{\beta}\frac{(\tau-a)^{\nu}\varphi(\tau)\,d\tau}{\tau-x}$$

$$I_{x\beta}{}^{\nu}I_{\alpha\tau}{}^{-\nu}\Phi \equiv \cos(\nu\pi)\,\Phi(x) + \sin(\nu\pi)\frac{(b-x)^{\nu}}{\pi}\int_{\alpha}^{\beta}\frac{\Phi(\tau)}{(b-\tau)^{\nu}}\frac{d\tau}{\tau-x} \qquad (2.2)$$

$$\left(I_{x\beta}{}^{\nu}\varphi \equiv \frac{1}{\Gamma(\nu)}\int_{x}^{\beta}\frac{\varphi(\tau)\,d\tau}{(\tau-x)^{1-\nu}},\quad \nu = 1-\mu\right)$$

Equations (2.2) allow us to arrive at an equation with Cauchy kernel which for $T=0$ is solvable in closed form. They also make it possible to eliminate (where necessary) the singular integrals with the aid of which the solution of the equation with Cauchy kernel is expressed. In conclusion we note that Eq. (1.2) is a particular case of Eq. (2.1) with $c_1 = c_2 = \sqrt{{}^1/_2\,a}$, $\mu = {}^1/_2$ and

$$T\varphi \equiv \int_{\alpha}^{x}\frac{\sqrt{2a+x-\tau}-\sqrt{2a}}{\sqrt{x-\tau}}\varphi(\tau)\,d\tau \equiv \int_{\alpha}^{x}\frac{\sqrt{x-\tau}}{\sqrt{2a}+\sqrt{2a+x-\tau}}\varphi(\tau)\,d\tau$$

BIBLIOGRAPHY

1. Sedov, L. I., Ploskie zadachi gidrodinamiki i aerodinamiki (Two-Dimensional Problems of Hydrodynamics and Aerodynamics), 2nd ed., Izd. Nauka, 1966.
2. Sedov, L. I., Prilozhenie teorii funktsii kompleksnogo peremennogo k nekotorym zadacham ploskoi gidrodinamiki (The application of the theory of functions to certain problems of two-dimensional hydrodynamics), Uspekhi mat. Nauk, No. 6, 1939.
3. Sakaliuk, K. D., Obobshchennoe integral'noe uravnenie Abelia (The generalized Abel integral equation), Dokl. Akad. Nauk SSSR, Vol. 131, No. 4, 1960.
4. Popov, G. Ia., Nekotorye svoistva klassicheskikh mnogochlenov i ikh primeneniia k kontaktnym zadacham (Some properties of classical polynomials and their applications to contact problems), PMM, Vol. 27, No. 5, 1963.
5. Arutiunian, N. Kh., Ploskaia kontaktnaia zadacha teorii plastichnosti so stepennym uprochneniem materiala (The two-dimensional contact problem of the theory of plasticity with power law strain hardening of the material), Izv. Akad. Nauk Arm. SSR, Ser. fiz.-Mat. Vol. 12, No. 2, 1959.
6. Arutiunian, N. Kh., Ploskaia kontaktnaia zadacha teorii polzuchesti (The two-dimensional contact problem of the theory of creep), PMM, Vol. 23, No. 5, 1959.
7. Arutiunian, N. Kh., Manukian, M. M., Kontaktnaia zadacha teorii

polzuchesti s uchetom sil treniia (The contact problem in the theory of creep with frictional forces taken into account), PMM, Vol. 27, No. 5, 1963.

8. Sakaliuk, K. D., Obobshchennoe integral'noe uravnenie Abelia s vnutrennimi koeffitsientami (The generalized Abel integral equation with internal coefficients). Uch. Zap. Kishinev. Univ. Vol. 82, 1965.

9. Gradshtein, I. S. and Ryzhik, I. M., Tablitsy integralov, summ, riadov, i proizvedenii (Tables of Integrals, Sums, Series and Products), Fizmatgiz, 1963.

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